P.h. Sem - II, 20.11.2024

Topology

Theorem — Let (X, J) be a topological space. A family B of subsets of X is a base for J iff $B^* = J$

Proof — Necessity

Let B be a base for J, then $b \subseteq J$ and each member G of J is a union of members of B. ie $G \in B^*$

Thus $J \subseteq B^*$ —— (1)

Also if $H \subset B^*$. then H is a union of members from B and since $B \subseteq J$, it follows that H is a union of members from J. Since J is a topology, we have $H \in J$

Thus $B^* \subseteq J$ —— (2)

From (1) and (2) we get

$$B^* = J$$

Sufficient

if $b^* = J$ then for each $G \in J$, $G \in B^*$ and hence G is a union of members from B.

Since

$$B \subseteq B^* = J$$

it follows that B is a base for J

Example The family $B = \{\{a\}, \{b\}, X\}$ is a base for the topology

$$J = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \text{ on}$$

$$X = \{a, b, c\}$$